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Extending the Black Hole Uniqueness Theorems II. Superstring Black Holes

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Abstract

We make use of an internal symmetry of a truncation of the bosonic sector of the superstring and $N = 4$ supergravity theories to write down an analogue of Robinson's identity for the black holes of this theory. This allows us to prove the uniqueness of a restricted class of black hole solutions. In particular, we can apply the methods of the preceding paper to prove the uniqueness of a class of accelerating black holes (the Stringy Ernst solution and Stringy C-metric) which incorporate the possibility of the black hole accelerating within an electromagnetic flux tube. These solutions and their associated uniqueness may be useful in future instanton calculations.

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I. SUPERSTRING BLACK HOLE UNIQUENESS THEOREMS

A. Introduction

In this paper we extend the black hole uniqueness theorems to the Superstring and $N = 4$ Supergravity theories. We will be making use of the notations and conventions in [1], where we proved the uniqueness of a class of accelerating black holes in Einstein-Maxwell theory. In order to make progress in the string theory we will be studying we will need to impose staticity rather than merely stationarity of the solutions, and naturally require the invariance of the dilaton under the action of the isometries generated by the Killing vectors. In addition we will only consider the case where the axionic field has been set equal to zero. This is consistent if we assume the electric and magnetic components are actually derived from two separate $U(1)$ gauge fields. The essential point to notice in our proof is that the effective Lagrangian in such a theory can be written as the sum of two copies of that which we find for pure gravity. We will need to verify that the Weyl coordinate system may be introduced and then make use of Robinson's identity to establish the uniqueness result.

Firstly we will establish the uniqueness of a class of black holes obtained by performing an internal symmetry (the Double Ehlers' transform) to a spherically symmetric solution found by Gibbons [2]. These solutions are asymptotically Melvin's Stringy Universe, it thus generalizes the result of Hiscock [3] for the Einstein-Maxwell theory. We could equally apply the theory to asymptotically flat solutions but one might feel that the uniqueness of such solutions should be proved under less stringent hypotheses, in particular Masood-ul-Alam has already proved the uniqueness of an asymptotically flat black hole solution in these theories [4].

Secondly, we turn to the Ernst solution and the C-metric, or rather their stringy variants and proceed to prove a theorem establishing their uniqueness (see for the proof in Einstein-Maxwell theory [1]). The solutions found here represent a generalization of those discussed by Dowker *et al.* [5], and reduce to them when the Double Ehlers' transform has equal parameters. It might be noted that they do not agree with those previously proposed by Ross [6].

In Sect. II we introduce the spherically symmetric solution in string theory that is the analogue of the Reissner-Nordstrøm black hole. We then perform a double Ehlers' transform to generate a new solution that will be the object of our uniqueness theorem. In the following section, Sect. III, we carefully state the hypotheses we need to prove the theorem and justify the introduction of Weyl coordinates by proving that the norm of the Killing bivector is a harmonic function on the relevant orbit space.

In Sect. III A we explain how Robinson's identity for the pure gravity can be exploited to give us a tool for establishing a uniqueness theorem in string theory and $N = 4$ supergravity subject to the hypotheses laid out in Sect. III. We then complete the proof of our theorem by presenting sufficient boundary conditions to make the appropriate boundary integral vanish. These conditions are laid out in Sect. III B.

Having demonstrated how we may establish a uniqueness theorem in these theories we go on to apply our methods to the Stringy C-metric and Stringy Ernst solution. The Stringy C-metric is that found by Dowker *et al.* [5]. We apply the double Ehlers' transformation to derive the Stringy Ernst solution. As in [1] we transform coordinates to ones which have a strong relationship to the elliptic functions and integrals. This is set out in Sect. IV. Then in Sect. IV A we write down the relevant boundary conditions to complete the uniqueness theorem for these solutions. Finally in the conclusion, Sect. IV B, we make a few comments on the difficulties in generalizing the result.

B. The $N = 4$ Supergravity and Superstring Theories

Let us turn to a truncated theory arising from the bosonic sector of the $N = 4$ Supergravity and Superstring Theories. These theories possesses a dilaton with coupling parameter equal to unity, as well as electric and magnetic potentials. For simplicity we will restrict attention to the static truncation of the harmonic map. The $N = 4$ theory possesses an axionic field, and six $U(1)$ gauge fields that combined have an $SO(6)$ invariance. Together with a suitable duality rotation it is possible to reduce the theory to one with just two $U(1)$ gauge fields, one purely electric, the other purely magnetic. At this point the axion decouples and can be consistently set equal to zero. What remains can be written in terms of an effective single electromagnetic field (with both electric and magnetic parts), see Gibbons [2] for further details. The Lagrangian density can then be written:

$$\mathcal{L} = \sqrt{|g|}\left(R - 2|\nabla\phi|^2 - e^{-2\phi}F_{ab}F^{ab}\right) \ . \tag{1.1}$$

After a dimensional reduction on a spacelike axial Killing vector field $m = \partial/\partial\varphi$ the density takes the form:

$$\mathcal{L} = \sqrt{\gamma}\left({}^3R - 2\left(\frac{|\nabla X|^2}{4X^2} + |\nabla\phi|^2 + \frac{e^{-2\phi}|\nabla\psi_e|^2}{X} + \frac{e^{2\phi}|\nabla\psi_m|^2}{X}\right)\right) \tag{1.2}$$

where

$$\boldsymbol{g} = X\boldsymbol{d}\varphi\otimes\boldsymbol{d}\varphi + X^{-1}\gamma_{ij}\boldsymbol{d}x^i\otimes\boldsymbol{d}x^j, \tag{1.3}$$

$$\boldsymbol{d}\psi_e = -i_m\boldsymbol{F}, \tag{1.4}$$

$$\boldsymbol{d}\psi_m = e^{-2\phi}i_m{*}\boldsymbol{F}, \tag{1.5}$$

${}^3R$ is the Ricci scalar of the metric γ_{ij}, and the metric γ_{ij} has been used to perform the contractions in Eq. (1.2). The Hodge dual in Eq. (1.5) is that from the four-dimensional metric (1.3). In order to derive Eq. (1.2) we have needed to perform a Legendre transform, which has the effect of changing the sign of the $|\nabla\psi_m|^2$ term from what one might have naïvely expected. We now define new coordinates

$$X_+ = X^{1/2}e^{\phi} \qquad \text{and} \qquad X_- = X^{1/2}e^{-\phi}. \tag{1.6}$$

Together with the electrostatic potentials $\psi_+ = \sqrt{2}\psi_e$ and $\psi_- = \sqrt{2}\psi_m$. The metric on the target space of the harmonic map is given by

$$G_{AB}\boldsymbol{d}\phi^A \otimes \boldsymbol{d}\phi^B = \frac{\boldsymbol{d}X_+ \otimes \boldsymbol{d}X_+ + \boldsymbol{d}\psi_+ \otimes \boldsymbol{d}\psi_+}{X_+^2} + \frac{\boldsymbol{d}X_- \otimes \boldsymbol{d}X_- + \boldsymbol{d}\psi_- \otimes \boldsymbol{d}\psi_-}{X_-^2}. \tag{1.7}$$

We remark that this precisely takes the form as the sum of two copies of the Lagrangian for pure gravity. For the moment we merely note that we can perform independent Ehlers' transformations to both X_+ and X_- to derive new solutions.

1. The Double Ehlers' Transformation

Performing independent Ehlers' transformations to the system yield the following:

$$X \mapsto \frac{X}{\left[1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)\right]\left[1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)\right]}; \tag{1.8}$$

$$e^{2\phi} \mapsto e^{2\phi}\frac{1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)}{1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)}; \tag{1.9}$$

$$\psi_+ \mapsto \frac{\psi_+ + \beta\left(Xe^{2\phi}+\psi_+^2\right)}{1+\beta^2\left(Xe^{2\phi}+\psi_+^2\right)}; \tag{1.10}$$

$$\psi_- \mapsto \frac{\psi_- + \gamma\left(Xe^{-2\phi}+\psi_-^2\right)}{1+\gamma^2\left(Xe^{-2\phi}+\psi_-^2\right)}. \tag{1.11}$$

In particular if we apply this to Minkowski space we generate the Stringy Melvin Universe:

$$\begin{aligned}\boldsymbol{g} = {} & \left(1+\beta^2 r^2\sin^2\theta\right)\left(1+\gamma^2 r^2\sin^2\theta\right)\left(-\boldsymbol{d}t\otimes\boldsymbol{d}t+\boldsymbol{d}r\otimes\boldsymbol{d}r+r^2\boldsymbol{d}\theta\otimes\boldsymbol{d}\theta\right)\\ & + \frac{r^2\sin^2\theta\,\boldsymbol{d}\varphi\otimes\boldsymbol{d}\varphi}{\left(1+\beta^2 r^2\sin^2\theta\right)\left(1+\gamma^2 r^2\sin^2\theta\right)};\end{aligned} \tag{1.12}$$

$$e^{2\phi} = \frac{1+\gamma^2 r^2\sin^2\theta}{1+\beta^2 r^2\sin^2\theta}; \tag{1.13}$$

$$\boldsymbol{A} = -\sqrt{2}\gamma r\cos\theta\,\boldsymbol{d}t + \frac{\beta r^2\sin^2\theta\,\boldsymbol{d}\varphi}{\sqrt{2}\left(1+\beta^2 r^2\sin^2\theta\right)}. \tag{1.14}$$

This solution represents the stringy generalization of Melvin's Universe. Whereas in Melvin's universe the electric and magnetic fields can be transformed into one another by a simple duality rotation without affecting the metric (meaning often that we need only consider a purely magnetic or electric universe), the stringy universe typically involves both electric and magnetic fields. These fields are parallel and provide a repulsive

force to counterbalance the attractive force of the spin zero dilaton and spin two graviton fields. The Stringy Melvin Universe will be important to us as it will model a strong electromagnetic field in string theory and one might be interested in the possible mediation of topological defects by such fields.

II. THE CLASS OF SOLUTIONS

Our starting point is the spherically symmetric solution found by Gibbons [2]:

$$\boldsymbol{g} = -\left(1-\frac{2M}{r}\right)\boldsymbol{dt}\otimes\boldsymbol{dt} + \left(1-\frac{2M}{r}\right)^{-1}\boldsymbol{dr}\otimes\boldsymbol{dr} + r\left(r-\frac{Q^2}{M}\right)\boldsymbol{d\Omega}^2. \tag{2.1}$$

The electromagnetic field and dilaton are given by

$$\boldsymbol{A} = \frac{Q}{r}\boldsymbol{dt}, \tag{2.2}$$

$$e^{2\phi} = 1 - \frac{Q^2}{Mr} \tag{2.3}$$

where we write ϕ for the dilaton field and φ for the angular coordinate.

Let us now apply the Double Ehlers' Transformation associated with the angular Killing vector $\partial/\partial\varphi$. The transformations are given by Eqs. (1.8) to (1.11).

The solution given above, Eqs. (2.1) to (2.2) have potentials:

$$X = r\left(r-\frac{Q^2}{M}\right)\sin^2\theta, \tag{2.4}$$

$$\psi_+ = 0, \tag{2.5}$$

$$\psi_- = \sqrt{2}Q\cos\theta, \tag{2.6}$$

together with (2.3). In consequence it is a simple matter to write down the transformed metric and fields:

$$\boldsymbol{g} = \Lambda\Theta\left(-\left(1-\frac{2M}{r}\right)\boldsymbol{dt}\otimes\boldsymbol{dt} + \left(1-\frac{2M}{r}\right)^{-1}\boldsymbol{dr}\otimes\boldsymbol{dr} + r\left(r-\frac{Q^2}{M}\right)\boldsymbol{d\theta}\otimes\boldsymbol{d\theta}\right) + \frac{r}{\Lambda\Theta}\left(r-\frac{Q^2}{M}\right)\sin^2\theta\boldsymbol{d\varphi}\otimes\boldsymbol{d\varphi}, \tag{2.7}$$

where

$$\Lambda = 1+\beta^2\left(r-\frac{Q^2}{M}\right)^2\sin^2\theta \qquad \text{and} \qquad \Theta = 1+\gamma^2r^2\sin^2\theta. \tag{2.8}$$

The new dilaton and potentials are given by

$$e^{2\phi} = \left(1 - \frac{Q^2}{M}\right) \frac{\Theta}{\Lambda}, \tag{2.9}$$

$$\psi_+ = \frac{\beta}{\Lambda}\left(r - \frac{Q^2}{M}\right)^2 \sin^2\theta, \tag{2.10}$$

$$\psi_- = \frac{1}{\Theta}\left[\sqrt{2}Q\cos\theta + \gamma\left(r^2\sin^2\theta + 2Q^2\cos^2\theta\right)\right]. \tag{2.11}$$

III. THE HYPOTHESES

Let us now list the hypotheses we will need to prove our uniqueness theorems:

- Axisymmetry: There exists a Killing vector m such that $\mathcal{L}_m \boldsymbol{g} = 0$, $\mathcal{L}_m \boldsymbol{F} = 0$ and $\mathcal{L}_m \phi = 0$ which generates a one-parameter group of isometries whose orbits are closed spacelike curves.
- Staticity: There exists a hypersurface orthogonal Killing vector field K such that $\mathcal{L}_K \boldsymbol{g} = 0$, $\mathcal{L}_K \boldsymbol{F} = 0$ and $\mathcal{L}_K \phi$ which generates a one-parameter group of isometries which acts freely and whose orbits near infinity are timelike curves.
- Commutivity: $[K, m] = 0$.
- Source-free Maxwell equations $\boldsymbol{dF} = 0$ and $\boldsymbol{\delta}\left(e^{-2\phi}\boldsymbol{F}\right) = 0$ together with the Einstein equations $R_{ab} = 2\nabla_a\phi\nabla_b\phi + 8\pi e^{-2\phi}T^{(F)}_{ab}$ where

$$T^{(F)}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \tag{3.1}$$

- The domain of outer communication is connected and simply-connected.
- The solution contains a single black hole.
- The solution is asymptotically the Stringy Melvin Universe.
- Boundary conditions (See section III B).

We remark that the Generalized Papapetrou theorem of as in [1] goes through with a few very minor changes to take account of the modified Einstein and Maxwell relations. In particular the invariance of the dilaton field under the symmetries reads

$$i_K \boldsymbol{d}\phi = 0 \qquad \text{and} \qquad i_m \boldsymbol{d}\phi = 0. \tag{3.2}$$

Let us define $\boldsymbol{T}(\boldsymbol{k}) = \left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right) k^a \boldsymbol{e}^b$ be the trace-reverse energy-momentum 1-form with respect to $\boldsymbol{k}$. Accordingly the dilaton does not contribute to $\boldsymbol{T}(\boldsymbol{k})$. In addition the Staticity condition means that the cross term in the metric vanishes, i.e., $W = 0$.

The next step is to introduce Weyl coordinates. We show that ρ is a harmonic function on the space of orbits. Explicitly we have $\rho^2 = XV$. Defining

$$(h_{AB}) = \begin{pmatrix} -V & 0 \\ 0 & X \end{pmatrix} \qquad \text{and} \qquad \left(h^{AB}\right) = \frac{1}{\rho^2}\begin{pmatrix} -X & 0 \\ 0 & V \end{pmatrix}, \tag{3.3}$$

we need to calculate

$${}^4R_{AB}h^{AB} = -\frac{1}{2\rho}\nabla^\alpha\left(\rho h^{AB}\nabla_\alpha h_{AB}\right) = -\frac{1}{\rho}\nabla^2\rho. \tag{3.4}$$

Here A and B refer to the t and φ coordinates whilst the covariant derivatives are with respect to the induced metric on the two-dimensional orbit space. Defining

$$E_\alpha = F_{t\alpha} \qquad \text{and} \qquad B_\alpha = F_{\varphi\alpha} \tag{3.5}$$

we have

$${}^4R_{tt} = e^{-2\phi}\left(2\boldsymbol{E.E} + \tfrac{1}{2}VF^2\right), \tag{3.6}$$

$${}^4R_{\varphi\varphi} = e^{-2\phi}\left(2\boldsymbol{B.B} - \tfrac{1}{2}XF^2\right). \tag{3.7}$$

where

$$F^2 = 2(-X\boldsymbol{E.E} + V\boldsymbol{B.B})\rho^{-2}. \tag{3.8}$$

Notice that the invariance of ϕ means that $\partial\phi/\partial t = 0$ and $\partial\phi/\partial\varphi = 0$, and that therefore $\nabla_A\phi\nabla_B\phi$ makes no contribution to ${}^4R_{AB}$. The result is

$$-\frac{1}{\rho}\nabla^2\rho = \frac{1}{\rho^2}\left(-{}^4R_{tt}X + {}^4R_{\varphi\varphi}V\right) = 0. \tag{3.9}$$

Thus ρ is harmonic and we may go on to introduce its harmonic conjugate, z together with t and φ that provide a coordinate system for the spacetime.

A. The Divergence Identity

We recall at this point our discussion in Sect. I B and in particular that the effective two dimensional Lagrangian arising from string theory and $N = 4$ Supergravity takes the form

$$\mathcal{L} = \rho\sqrt{|\gamma|}\left[\frac{|\nabla X_+|^2 + |\nabla\psi_+|^2}{X_+^2} + \frac{|\nabla X_-|^2 + |\nabla\psi_-|^2}{X_-^2}\right] \tag{3.10}$$

where

$$X_+^2 = Xe^{2\phi} \qquad \text{and} \qquad X_-^2 = Xe^{-2\phi}. \tag{3.11}$$

Each term in the above Lagrangian is a copy of the Lagrangian for pure gravity and in consequence we may thus use Robinson's identity [7],

$$\begin{aligned} \nabla . \left(\rho \nabla \left(\frac{\hat{X_+}^2 + \hat{\psi_+}^2}{X_+^{(1)} X_+^{(2)}} + \frac{\hat{X_-}^2 + \hat{\psi_-}^2}{X_-^{(1)} X_-^{(2)}} \right) \right) & \\ = F(X_+^{(1)}, X_+^{(2)}, \psi_+^{(1)}, \psi_+^{(2)}) + F(X_-^{(1)}, X_-^{(2)}, \psi_-^{(1)}, \psi_-^{(2)}) \geq 0, & \end{aligned} \tag{3.12}$$

where $F(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)})$ is defined by

$$\begin{aligned} F(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)}) &= \frac{\rho}{X^{(1)} X^{(2)}} \left[\frac{\hat{Y} \nabla Y^{(1)}}{X^{(1)}} + \frac{X^{(1)} \nabla X^{(2)}}{X^{(2)}} - \nabla X^{(1)} \right]^2 \\ &+ \frac{\rho}{X^{(1)} X^{(2)}} \left[\frac{\hat{Y} \nabla Y^{(2)}}{X^{(2)}} - \frac{X^{(2)} \nabla X^{(1)}}{X^{(1)}} + \nabla X^{(2)} \right]^2 \\ &+ \frac{\rho}{2X^{(1)} X^{(2)}} \left[\left(\frac{\nabla Y^{(2)}}{X^{(2)}} - \frac{\nabla Y^{(1)}}{X^{(1)}} \right) (X^{(1)} + X^{(2)}) - \left(\frac{\nabla X^{(2)}}{X^{(2)}} + \frac{\nabla X^{(1)}}{X^{(1)}} \right) \hat{Y} \right]^2 \\ &+ \frac{\rho}{2X^{(1)} X^{(2)}} \left[\left(\frac{\nabla Y^{(2)}}{X^{(2)}} + \frac{\nabla Y^{(1)}}{X^{(1)}} \right) \hat{X} - \left(\frac{\nabla X^{(2)}}{X^{(2)}} + \frac{\nabla X^{(1)}}{X^{(1)}} \right) \hat{Y} \right]^2 . \end{aligned} \tag{3.13}$$

We have defined $\hat{A} = A_2 - A_1$ etc. It is now evident that we may use this divergence identity to provide us with the key tool in establishing a black hole uniqueness theorem. To complete the proof we will want to change coordinates, and impose suitable boundary conditions to make the relevant boundary integral vanish. We make the change of coordinates:

$$\rho = r \sin \theta, \tag{3.14}$$

$$z = r \cos \theta. \tag{3.15}$$

The value of r runs from M to infinity (we adjust the additive constant to z to make the horizon run from $-M \leq z \leq M$). The overall scaling of ρ and z is made such that asymptotically r becomes the radial coordinate of the Stringy Melvin Universe, i.e.,

$$\boldsymbol{g} \sim A\rho^4 (-\boldsymbol{d}t \otimes \boldsymbol{d}t + \boldsymbol{d}\rho \otimes \boldsymbol{d}\rho + \boldsymbol{d}z \otimes \boldsymbol{d}z) + \frac{1}{A\rho^2} \boldsymbol{d}\varphi \otimes \boldsymbol{d}\varphi. \tag{3.16}$$

with φ taking values in $[0, 2\pi)$. It is worth remarking that we cannot rescale the coordinates and parameters and retain this form whilst leaving the range of φ unchanged, except for the trivial instance of multiplying the coordinates by -1.

The two dimensional domain we work on is the semi-infinite rectangle, $r > M$ and $-\pi/2 \leq \theta \leq \pi/2$, and the boundary integral we require to vanish is now given by

$$\int r\cos\theta \left(r d\theta \frac{\partial}{\partial r} - \frac{dr}{r}\frac{\partial}{\partial \theta}\right) \left(\frac{\hat{X_+}^2 + \hat{\psi}_+^2}{X_+^{(1)} X_+^{(2)}} + \frac{\hat{X_-}^2 + \hat{\psi}_-^2}{X_-^{(1)} X_-^{(2)}} \right) = 0. \tag{3.17}$$

B. Boundary Conditions

We now need to impose suitable boundary conditions to make the boundary integral vanish. The following prove to be sufficient. At infinity we require

$$X_+ = \frac{1}{\beta^2 \sin\theta}\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{3.18}$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial r} = -\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{3.19}$$

$$\psi_+ = \frac{1}{\beta} - \frac{1}{\beta^3 \sin^2\theta}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right); \tag{3.20}$$

$$\frac{\partial \psi_+}{\partial r} = \frac{2}{\beta^3 \sin^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right); \tag{3.21}$$

$$X_- = \frac{1}{\gamma^2 \sin\theta}\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{3.22}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial r} = -\frac{1}{r} + O\left(\frac{1}{r^2}\right); \tag{3.23}$$

$$\psi_- = \frac{1}{\gamma} - \frac{1 + \sqrt{2}\gamma Q \cos\theta}{\gamma^3 \sin^2\theta}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right); \tag{3.24}$$

$$\frac{\partial \psi_-}{\partial r} = \frac{2\left(1 + \sqrt{2}\gamma Q \cos\theta\right)}{\gamma^3 \sin^2\theta}\frac{1}{r^3} + O\left(\frac{1}{r^4}\right). \tag{3.25}$$

On the axes we require (setting $\mu = \sin\theta$)

$$\frac{1}{X_+}\frac{\partial X_+}{\partial \mu} = \frac{-\mu}{1-\mu^2} + O(1); \tag{3.26}$$

$$\frac{\partial X_+}{\partial r} = O\left(\left(1-\mu^2\right)^{1/2}\right); \tag{3.27}$$

$$\psi_+ = O(1-\mu^2); \tag{3.28}$$

$$\frac{\partial \psi_+}{\partial \mu} = O(1); \tag{3.29}$$

$$\frac{\partial \psi_+}{\partial r} = O(1-\mu^2); \tag{3.30}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial \mu} = \frac{-\mu}{1-\mu^2} + O(1); \tag{3.31}$$

$$\frac{\partial X_-}{\partial r} = O\left(\left(1-\mu^2\right)^{1/2}\right); \tag{3.32}$$

$$\psi_- = \frac{(\sqrt{2}\mu + 2\gamma Q)Q}{1+2\gamma^2 Q^2} + O(1-\mu^2); \tag{3.33}$$

$$\frac{\partial \psi_-}{\partial \mu} = O(1); \tag{3.34}$$

$$\frac{\partial \psi_-}{\partial r} = O(1-\mu^2); \tag{3.35}$$

where the boundaries correspond to $\mu = \pm 1$. On the horizon we require regularity of X_+, X_-, ψ_+ and ψ_-. These conditions are sufficient to make the boundary integral vanish and hence establish our uniqueness result.

IV. UNIQUENESS THEOREMS FOR THE STRINGY C-METRIC AND STRINGY-ERNST SOLUTION

In [1] we proved the uniqueness of both the C-metric and the Ernst solution. In this section we exploit the techniques developed there together with the string uniqueness formalism we have just been using to show that given any Stringy C-metric or Stringy Ernst solution then the boundary conditions uniquely specify the solution. Our philosophy here is slightly less ambitious than for Einstein-Maxwell theory; in the latter case we took the position that any candidate solution that resembled the Ernst solution at infinity was indeed an Ernst solution provided one of the quantities determined on the boundary was greater than a critical value. Here we assume we have an Ernst solution that does satisfy the boundary conditions and prove that no other solution can have the same boundary conditions.

Our starting point is the Dilaton C-metric found by Dowker *et al.* [5],

$$\boldsymbol{g} = \frac{1}{A^2(x-y)^2}\left[F(x)G(y)\boldsymbol{d}t\otimes\boldsymbol{d}t + \frac{F(y)\boldsymbol{d}x\otimes\boldsymbol{d}x}{G(y)} - \frac{F(x)\boldsymbol{d}y\otimes\boldsymbol{d}y}{G(y)} \right.$$
$$\left. + F(y)G(x)\boldsymbol{d}\varphi\otimes\boldsymbol{d}\varphi\right], \tag{4.1}$$

where

$$e^{-2\phi} = \frac{F(y)}{F(x)}, \tag{4.2}$$

$$\boldsymbol{A} = \sqrt{\frac{r_+ r_-}{2}}(x-x_2)\boldsymbol{d}\varphi, \tag{4.3}$$

$$F(\xi) = 1 + r_- A\xi, \tag{4.4}$$

$$G(\xi) = 1 - \xi^2 - r_+ A\xi^3. \tag{4.5}$$

We have labelled the roots of $G(x)$ as $x_3 < x_2 < x_1$ with $x_1 > 0$. The quantity x_4 corresponds to setting $F(x) = 0$, for which we assume $x_4 < x_3$ so as to represent an inner horizon for the black hole.

It is advantageous to represent this solution in terms of the Jacobi elliptic functions. We transform to new coordinates using

$$\frac{\chi}{M} = \int_{x_2}^{x} \frac{d\xi}{\sqrt{F(\xi)G(\xi)}} \qquad \text{and} \qquad \frac{\eta}{M} = \int_{y}^{x_2} \frac{d\xi}{\sqrt{-F(\xi)G(\xi)}}. \tag{4.6}$$

with $M = \sqrt{e_1 - e_3}$ where $e_i = \wp(\omega_i)$ and ω_i being a half period as we had in Appendix A of [1]. The appropriate invariants of the $\wp$-function are given by

$$g_2 = \frac{1 + 3A^2 r_-^2 - 9A^2 r_+ r_-}{12}, \tag{4.7}$$

$$g_3 = \frac{2 - 27A^2 r_+^2 - 18A^2 r_-^2 + 27A^2 r_+ r_- + 27A^4 r_+ r_-^3}{432}. \tag{4.8}$$

Writing the metric as

$$\boldsymbol{g} = -V\boldsymbol{d}t \otimes \boldsymbol{d}t + X\boldsymbol{d}\phi \otimes \boldsymbol{d}\phi + \Sigma\left(\boldsymbol{d}\chi \otimes \boldsymbol{d}\chi + \boldsymbol{d}\eta \otimes \boldsymbol{d}\eta\right), \tag{4.9}$$

we find:

$$X = \frac{4L^2\left(1 - D\,\mathrm{sn}^2\,\eta\right)\left(1 - E\,\mathrm{sn}^2\,\eta\right)\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi}{\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\eta + E\,\mathrm{sn}^2\,\eta\right)\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}; \tag{4.10}$$

$$V = \frac{4L^2\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi\right)\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\eta\,\mathrm{dn}^2\,\eta}{\left(1 - D\,\mathrm{sn}^2\,\eta\right)\left(1 - E\,\mathrm{sn}^2\,\eta\right)\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}; \tag{4.11}$$

$$\Sigma = \frac{16H^2L^2\left(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi\right)\left(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi\right)\left(1 - D\,\mathrm{sn}^2\,\eta\right)^2\left(1 - E\,\mathrm{sn}^2\,\eta\right)}{\kappa^2\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}. \tag{4.12}$$

We have written $M = AL$ together with

$$\kappa = \left.\frac{d(F(\xi)G(\xi))}{d\xi}\right|_{\xi=x_2}, \qquad D = \frac{1 + k'^2}{3} - \frac{1}{24M^2}\left.\frac{d^2(F(\xi)G(\xi))}{d\xi^2}\right|_{\xi=x_2} \tag{4.13}$$

and

$$E = D + \frac{r_- A\kappa}{4M^2 H}, \qquad H = 1 + Ar_- x_2. \tag{4.14}$$

The quantity ρ is given by

$$\rho = \frac{4L^2\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,\mathrm{sn}\,\eta\,\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{\left(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi\right)^2}. \tag{4.15}$$

Thus we have $z - i\rho = 2L^2\wp(\chi + i\eta)$. The dilaton and vector potential are given by the expressions

$$e^{-2\phi} = \frac{(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(1 - E\,\mathrm{sn}^2\,\eta)}{(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)\,(1 - D\,\mathrm{sn}^2\,\chi)}\,; \tag{4.16}$$

$$\boldsymbol{A} = \frac{QD\,\mathrm{sn}^2\,\chi\boldsymbol{d}\varphi}{4\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)}\,; \qquad Q = \frac{\kappa\sqrt{r_+ r_-}}{\sqrt{2}A^3L^2D}. \tag{4.17}$$

Performing the transformations Eqs. (1.8) to (1.11) we arrive at the metric of interest. The new metric and fields we have derived are:

$$X = \frac{4L^2\,(1 - D\,\mathrm{sn}^2\,\eta)\,(1 - E\,\mathrm{sn}^2\,\eta)\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi}{\Lambda\Theta\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(\mathrm{cn}^2\,\eta + E\,\mathrm{sn}^2\,\eta)\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}\,; \tag{4.18}$$

$$V = \frac{4L^2\Lambda\Theta\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)\,\mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\eta\,\mathrm{dn}^2\,\eta}{(1 - D\,\mathrm{sn}^2\,\eta)\,(1 - E\,\mathrm{sn}^2\,\eta)\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}; \tag{4.19}$$

$$\Sigma = \frac{16H^2L^2\Lambda\Theta\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)\,(1 - D\,\mathrm{sn}^2\,\eta)^2\,(1 - E\,\mathrm{sn}^2\,\eta)}{\kappa^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}; \tag{4.20}$$

with

$$\Lambda = 1 + \beta^2\left\{\frac{4L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1 - D\,\mathrm{sn}^2\,\eta)^2}{(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2} + \frac{Q^2D^2\,\mathrm{sn}^4\,\chi}{16\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)}\right\}; \tag{4.21}$$

$$\Theta = 1 + \frac{4\gamma^2L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1 - E\,\mathrm{sn}^2\,\eta)^2}{(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}\;. \tag{4.22}$$

The dilaton is given by

$$e^{-2\phi} = \frac{\Lambda\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(1 - E\,\mathrm{sn}^2\,\eta)}{\Theta\,(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)\,(1 - D\,\mathrm{sn}^2\,\chi)}\;. \tag{4.23}$$

We record the values of the quantities $X_\pm$ and the potentials $\psi_\pm$:

$$X_+ = \frac{2L\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,(1 - D\,\mathrm{sn}^2\,\eta)}{\Lambda\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)}; \tag{4.24}$$

$$X_- = \frac{2L\,\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi\,\mathrm{dn}\,\chi\,(1 - E\,\mathrm{sn}^2\,\eta)}{\Theta\,(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)}; \tag{4.25}$$

$$\begin{aligned}\psi_+ = {} & \frac{1}{\Lambda}\left\{\frac{QD\,\mathrm{sn}^2\,\chi}{4\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)}\right.\\ & \left.+\beta\left[\frac{4L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1 - D\,\mathrm{sn}^2\,\eta)^2}{(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2} + \frac{Q^2D^2\,\mathrm{sn}^4\,\chi}{16\,(\mathrm{cn}^2\,\chi + D\,\mathrm{sn}^2\,\chi)}\right]\right\}; \end{aligned} \tag{4.26}$$

$$\psi_- = \frac{4\gamma L^2\,\mathrm{sn}^2\,\chi\,\mathrm{cn}^2\,\chi\,\mathrm{dn}^2\,\chi\,(1 - E\,\mathrm{sn}^2\,\eta)^2}{\Theta\,(\mathrm{cn}^2\,\chi + E\,\mathrm{sn}^2\,\chi)^2\,(\mathrm{sn}^2\,\chi + \mathrm{sn}^2\,\eta\,\mathrm{cn}^2\,\chi)^2}. \tag{4.27}$$

We will be interested in the behaviour of the fields as one takes the limits $\chi \to 0$, $u \to 0$ with $u = K - \chi$ and $R \to \infty$. The appropriate boundary conditions we need to make the boundary integral vanish are presented in the next section.

A. Boundary Conditions for the Stringy Ernst Solution and C-Metric

In order to complete the proof of the uniqueness for the Stringy Ernst solution and Stringy C-metric it only remains to write down a set of boundary conditions that will make the boundary integral vanish. It is fairly simple to verify that the conditions given in the following two subsections are sufficient for this purpose.

1. Boundary Conditions for the Stringy Ernst Solution Uniqueness Theorem

To start with we will require all the fields to be regular (and in addition for X_+ and X_- to not vanish) as one approaches the acceleration and event horizons. Near the axis $\chi = 0$ we demand

$$\frac{1}{X_+}\frac{\partial X_+}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{4.28}$$

$$\frac{\partial X_+}{\partial \eta} = O\left(\chi\right); \tag{4.29}$$

$$\psi_+ = O\left(\chi^2\right); \tag{4.30}$$

$$\frac{\partial \psi_+}{\partial \chi} = O\left(\chi\right); \tag{4.31}$$

$$\frac{\partial \psi_+}{\partial \eta} = O\left(\chi\right); \tag{4.32}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial \chi} = \frac{1}{\chi} + O(1); \tag{4.33}$$

$$\frac{\partial X_-}{\partial \eta} = O\left(\chi\right); \tag{4.34}$$

$$\psi_- = O\left(\chi^2\right); \tag{4.35}$$

$$\frac{\partial \psi_-}{\partial \chi} = O\left(\chi\right); \tag{4.36}$$

$$\frac{\partial \psi_-}{\partial \eta} = O\left(\chi\right). \tag{4.37}$$

For the other axis we will require

$$\frac{1}{X_+}\frac{\partial X_+}{\partial u} = \frac{1}{u} + O(1); \tag{4.38}$$

$$\frac{\partial X_+}{\partial \eta} = O\left(u\right); \tag{4.39}$$

$$\psi_+ = \frac{Q\left(4+\beta Q\right)}{16+\beta^2 Q^2} + O\left(u^2\right); \tag{4.40}$$

$$\frac{\partial \psi_+}{\partial u} = O\left(u^2\right); \tag{4.41}$$

$$\frac{\partial \psi_+}{\partial \eta} = O\left(u\right); \tag{4.42}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial u} = \frac{1}{u} + O(1); \tag{4.43}$$

$$\frac{\partial X_-}{\partial \eta} = O\left(u\right); \tag{4.44}$$

$$\psi_- = O\left(u^2\right); \tag{4.45}$$

$$\frac{\partial \psi_-}{\partial u} = O\left(u\right); \tag{4.46}$$

$$\frac{\partial \psi_-}{\partial \eta} = O\left(u\right). \tag{4.47}$$

Whilst as $R \to \infty$ with $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$ we will demand

$$X_+ = \frac{1}{2\beta^2 L \sin\theta}\,\frac{1}{R^{1/2}} + O\left(\frac{1}{R^{3/2}}\right); \tag{4.48}$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial R} = -\frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{4.49}$$

$$\frac{\partial X_+}{\partial \theta} = O\left(\frac{1}{R^{1/2}}\right); \tag{4.50}$$

$$\psi_+ = \frac{1}{\beta} - \frac{1}{4\beta^3 L^2 \sin^2\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{4.51}$$

$$\frac{\partial \psi_+}{\partial R} = \frac{1}{4\beta^3 L^2 \sin^2\theta}\,\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{4.52}$$

$$\frac{\partial \psi_+}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{4.53}$$

$$X_- = \frac{1}{2\gamma^2 L \sin\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{4.54}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial R} = -\frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{4.55}$$

$$\frac{\partial X_-}{\partial \theta} = O\left(\frac{1}{R^{1/2}}\right); \tag{4.56}$$

$$\psi_- = \frac{1}{\gamma} - \frac{1}{4\gamma^3 L^2 \sin^2\theta}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{4.57}$$

$$\frac{\partial\psi_-}{\partial R} = \frac{1}{4\sqrt{2}\gamma^3 L^2 \sin^2\theta}\,\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{4.58}$$

$$\frac{\partial\psi_-}{\partial\theta} = O\left(\frac{1}{R^{1/2}}\right). \tag{4.59}$$

These boundary conditions are sufficient to establish the uniqueness of the Stringy Ernst solutions. For good measure we also present the boundary conditions for the Stringy C-metric problem.

2. Boundary Conditions for the Stringy C-Metric Uniqueness Theorem

The appropriate conditions are as follows. Near $\chi = 0$ we will insist

$$\frac{1}{X_+}\frac{\partial X_+}{\partial\chi} = \frac{1}{\chi} + O(1); \tag{4.60}$$

$$\frac{\partial X_+}{\partial\eta} = O(\chi); \tag{4.61}$$

$$\psi_+ = O\left(\chi^2\right); \tag{4.62}$$

$$\frac{\partial\psi_+}{\partial\chi} = O(\chi); \tag{4.63}$$

$$\frac{\partial\psi_+}{\partial\eta} = O(\chi); \tag{4.64}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial\chi} = \frac{1}{\chi} + O(1); \tag{4.65}$$

$$\frac{\partial X_-}{\partial\eta} = O(\chi); \tag{4.66}$$

$$\psi_- = O\left(\chi^2\right); \tag{4.67}$$

$$\frac{\partial\psi_-}{\partial\chi} = O(\chi); \tag{4.68}$$

$$\frac{\partial\psi_-}{\partial\eta} = O(\chi). \tag{4.69}$$

For the other axis we will require

$$\frac{1}{X_+}\frac{\partial X_+}{\partial u} = \frac{1}{u} + O(1); \tag{4.70}$$

$$\frac{\partial X_+}{\partial\eta} = O(u); \tag{4.71}$$

$$\psi_+ = \frac{Q}{2\sqrt{2}} + O\left(u^2\right); \tag{4.72}$$

$$\frac{\partial\psi_+}{\partial u} = O(u); \tag{4.73}$$

$$\frac{\partial \psi_+}{\partial \eta} = O\left(u\right); \tag{4.74}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial u} = \frac{1}{u} + O(1); \tag{4.75}$$

$$\frac{\partial X_-}{\partial \eta} = O\left(u\right); \tag{4.76}$$

$$\psi_- = O\left(u^2\right); \tag{4.77}$$

$$\frac{\partial \psi_-}{\partial u} = O\left(u\right); \tag{4.78}$$

$$\frac{\partial \psi_-}{\partial \eta} = O\left(u\right). \tag{4.79}$$

Whilst as $R \to \infty$ with $\chi = R^{-1/2}\sin\theta$ and $\eta = R^{-1/2}\cos\theta$ we will demand

$$X_+ = 2L\sin\theta\, R^{1/2} + O\left(1\right); \tag{4.80}$$

$$\frac{1}{X_+}\frac{\partial X_+}{\partial R} = \frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{4.81}$$

$$\frac{\partial X_+}{\partial \theta} = O\left(R^{1/2}\right); \tag{4.82}$$

$$\psi_+ = \frac{QD\sin^2\theta}{2\sqrt{2}}\,\frac{1}{R} + O\left(\frac{1}{R^2}\right); \tag{4.83}$$

$$\frac{\partial \psi_+}{\partial R} = -\frac{QD\sin^2\theta}{2\sqrt{2}}\,\frac{1}{R^2} + O\left(\frac{1}{R^3}\right); \tag{4.84}$$

$$\frac{\partial \psi_+}{\partial \theta} = O\left(\frac{1}{R}\right); \tag{4.85}$$

$$X_- = 2L\sin\theta\, R^{1/2} + O\left(1\right); \tag{4.86}$$

$$\frac{1}{X_-}\frac{\partial X_-}{\partial R} = \frac{1}{2R} + O\left(\frac{1}{R^2}\right); \tag{4.87}$$

$$\frac{\partial X_-}{\partial \theta} = O\left(R^{1/2}\right); \tag{4.88}$$

$$\psi_- = O\left(\frac{1}{R^2}\right); \tag{4.89}$$

$$\frac{\partial \psi_-}{\partial R} = O\left(\frac{1}{R^3}\right); \tag{4.90}$$

$$\frac{\partial \psi_-}{\partial \theta} = O\left(\frac{1}{R}\right). \tag{4.91}$$

B. Conclusion

We have been able to prove the uniqueness of two classes of asymptotically Melvin black holes. We would hope that the formalism developed in this chapter to prove the uniqueness of our class of black holes could be used to prove the uniqueness of other classes of static solutions in these theories. We would also like to have a formalism that incorporates the possibility of rotation and includes the axionic field, however it seems likely that such an extension would not be straightforward. The crux of the uniqueness proof is to establish the positivity of a suitable divergence. It turned out that for the static truncation of string theory that we considered the Lagrangian split into two separate copies of that for pure gravity. Consequently we could simply add together two copies of the relevant divergence identity (Robinson's identity) to furnish us with an expression that we could use in our black hole uniqueness investigations. If we include rotation or an axionic field the Lagrangian will not decompose so easily, and we would need to deal with it as a whole. This is problematical as the target space of the harmonic map possesses (at least) two timelike directions. Unfortunately this prohibits a simple application of the Mazur construction or a suitable analogue of the construction presented in [1]. It seems that Bunting's approach may be the best way forward under these circumstances relying, as it does, more heavily on the negative curvature of the target space metric than on its particular form as an $SU(1,2)/S\left(U(1)\times U(2)\right)$ symmetric space harmonic mapping system.

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